

CROSSING CHANGE ALTERNATING KNOTS

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ABSTRACT. In this paper we define Crossing Change Alternating Knots (CCA knots) and their generalization: k -CCA knots.

Definition 0.1. *Let be given a diagram D of a knot (or link). In D we make a crossing change in every crossing separately, and the rest of the crossings remain unchanged. From the diagram D with n crossings we obtain n new diagrams, each with a single crossing changed, and the corresponding n knots (or links). A diagram D is called Crossing Change Alternating (shortly, CCA) if all the knots (links) obtained by the crossing changes are alternating. A knot (or link) K is CCA if it has at least one CCA diagram.*

It is clear that a CCA knot (or link) could be alternating, or non-alternating.

If an alternating knot has a minimal CCA diagram, all its minimal diagrams are CCA (according to Tait Flying Theorem). A large class of CCA knots and links are rational knots and links.

In the case of alternating knots (links) it is sufficient to find one minimal diagram which is CCA, and all its minimal diagrams will be CCA. However, this is not true for non-alternating knots (links): a non-alternating knot (link) can have two different minimal diagrams, where one is CCA, and the other is not. For example, the minimal diagram $(2\,1, 2)(3, -2)$ (Fig. 1a) of the knot 10_{150} is not CCA, but its another minimal diagram $8^* - 2 : .20 : . - 1. - 1$ is CCA.

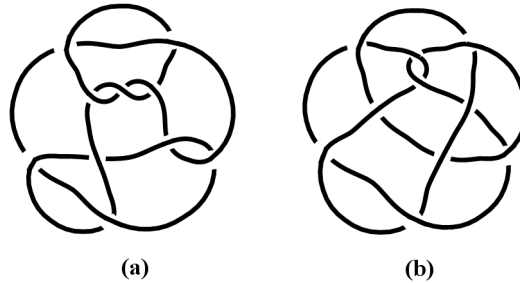


FIGURE 1. (a) The minimal not CCA diagram $(2\,1, 2)(3, -2)$ of the knot 10_{150} ; (b) the minimal CCA diagram $8^* - 2 : .20 : . - 1. - 1$ of the same knot.

Moreover, CCA-property is not necessarily realized on minimal diagrams. For example, all minimal diagrams of the knot 10_{151} (given in Conway notation as $(2\,1, 2)(2\,1, -2)$) are not CCA, but its non-minimal diagram $6^*2 - 1 - 1.2 : 2\,0$ is CCA, so the knot 10_{151} is a CCA knot without minimal CCA diagrams.

Thanks to the last example, proving that some knot is CCA is very difficult, because we need to check all diagrams, and not just the minimal ones. As the obstruction for a knot to be CCA we can use the alternation number. The *alternation number* of a link L , denoted by $alt(L)$, is the minimal number of crossing changes needed to deform L into an alternating link, where the minimum is taken over all diagrams of L [1]. According to T. Abe [2], for

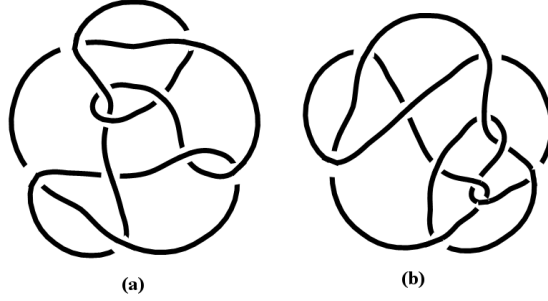


FIGURE 2. (a) The minimal not CCA diagram $(21, 2) (21, -2)$ of the knot 10_{151} . All its minimal diagrams are not CCA; (b) non-minimal CCA diagram $6^*2 - 1 - 1.2 : 20$ of the same knot.

every knot K alternation number satisfies the inequality $|\frac{s(K) - (-\sigma(K))}{2}| \leq alt(K)$, where $s(K)$ is the Rasmussen signature of K , and $-\sigma(K)$ the negative signature of K . It is clear that any knot K with $alt(K) > 1$ cannot be CCA. T. Abe [2] proved that for every torus knot $T_{p,q}$ ($2 \leq p < q$)

- (1) $alt(T_{p,q}) = 0 \Leftrightarrow p = 2$;
- (2) $alt(T_{p,q}) = 1 \Leftrightarrow (p, q) = (3, 4) \text{ or } (3, 5)$;
- (3) $alt(T_{p,q}) \geq 2 \Leftrightarrow \text{otherwise}$.

Hence, we know that there is an infinite number of knots that are not CCA. However, the obstruction $|\frac{s(K) - (-\sigma(K))}{2}| \leq alt(K)$ is not strong enough for many knots for which we suspect that are not CCA. E.g., for every alternating knot the Rasmussen signature and signature coincide, and there are many alternating knots which are candidates for knots that are not CCA. Such candidates are all alternating knots with a minimal diagram which is not CCA.

Definition 1 can be generalized in order to define k -CCA knots:

Definition 0.2. Let be given a diagram D of a knot (or link). In D we make k crossing changes in each subset of crossings consisting from k crossings ($1 \leq k \leq \lfloor \frac{n}{2} \rfloor + 1$), and the rest of the crossings remain unchanged. From the diagram D with n crossings we obtain $\binom{n}{k}$ new diagrams, each with k crossings changed, and the corresponding $\binom{n}{k}$ knots (or links). A diagram D is called k -Crossing Change Alternating (shortly, k -CCA) if all the knots (links) obtained by the crossing changes are alternating. A knot (or link) K is k -CCA if it has at least one k -CCA diagram.

In the same way as before, we expect that there exist knots that are k -CCA, but without a minimal k -CCA diagram, so it will be very difficult to conclude that some knot is k -CCA or not. As the obstruction for a knot to be k -CCA we can use the same obstruction as before. However, based on the computations on minimal diagrams, there will be many candidates for knots that are not k -CCA, for which will be very difficult to show that they are not k -CCA. Making computations only on the minimal diagrams, we can conclude that, e.g., the diagram $21, 21, 2+$ of the knot 9_{28} is not 1-CCA, it is 2-CCA, and not 3-, 4-, nor 5-CCA. On the other hand, the minimal diagram $21, 21, -2$ of the knot 8_{20} is 1-, 2-, and 4-CCA, but is not 3-CCA. After checking all minimal diagrams of this knot, we can conclude that none of them is 3-CCA, but we are not able to say that the knot $21, 21, -2$ is 3-CCA or not, because we need to check all its non-minimal diagrams, and for this knot the mentioned obstruction based on alternating number is not helpful.

1. CCA KNOTS WITH $n \leq 12$ CROSSINGS

All computations in this paper are made in the program *LinKnot* [3].

The first table contains alternating CCA knots with $n \leq 12$ crossings with minimal CCA diagrams.

3 ₁	4 ₁	5 ₁	5 ₂	6 ₁	6 ₂	6 ₃	7 ₁	7 ₂	7 ₃
7 ₄	7 ₅	7 ₆	7 ₇	8 ₁	8 ₂	8 ₃	8 ₄	8 ₅	8 ₆
8 ₇	8 ₈	8 ₉	8 ₁₀	8 ₁₁	8 ₁₂	8 ₁₃	8 ₁₄	8 ₁₅	8 ₁₆
8 ₁₇	8 ₁₈	9 ₁	9 ₂	9 ₃	9 ₄	9 ₅	9 ₆	9 ₇	9 ₈
9 ₉	9 ₁₀	9 ₁₁	9 ₁₂	9 ₁₃	9 ₁₄	9 ₁₅	9 ₁₇	9 ₁₈	9 ₁₉
9 ₂₀	9 ₂₁	9 ₂₂	9 ₂₃	9 ₂₅	9 ₂₆	9 ₂₇	9 ₂₉	9 ₃₀	9 ₃₁
9 ₃₄	9 ₃₅	9 ₃₆	9 ₃₇	9 ₃₈	9 ₃₉	9 ₄₁	10 ₁	10 ₂	10 ₃
10 ₄	10 ₅	10 ₆	10 ₇	10 ₈	10 ₉	10 ₁₀	10 ₁₁	10 ₁₂	10 ₁₃
10 ₁₄	10 ₁₅	10 ₁₆	10 ₁₇	10 ₁₈	10 ₁₉	10 ₂₀	10 ₂₁	10 ₂₂	10 ₂₃
10 ₂₄	10 ₂₅	10 ₂₆	10 ₂₇	10 ₂₈	10 ₂₉	10 ₃₀	10 ₃₁	10 ₃₂	10 ₃₃
10 ₃₄	10 ₃₅	10 ₃₆	10 ₃₇	10 ₃₈	10 ₃₉	10 ₄₀	10 ₄₁	10 ₄₂	10 ₄₃
10 ₄₄	10 ₄₅	10 ₄₆	10 ₄₇	10 ₅₀	10 ₅₁	10 ₅₂	10 ₅₃	10 ₅₄	10 ₅₅
10 ₅₈	10 ₅₉	10 ₆₀	10 ₆₁	10 ₆₂	10 ₆₃	10 ₆₇	10 ₆₈	10 ₆₉	10 ₇₆
10 ₇₇	10 ₇₈	10 ₇₉	10 ₈₂	10 ₈₄	10 ₈₅	10 ₈₇	10 ₉₀	10 ₉₃	10 ₁₀₀
10 ₁₀₂	10 ₁₀₃	10 ₁₀₄	10 ₁₀₆	10 ₁₀₈	10 ₁₀₉	10 ₁₁₀	10 ₁₁₁	10 ₁₁₂	10 ₁₁₄
10 ₁₁₈	10 ₁₁₉	10 ₁₂₀	10 ₁₂₃	K11a4	K11a8	K11a9	K11a10	K11a11	K11a12
K11a13	K11a14	K11a15	K11a21	K11a33	K11a35	K11a37	K11a39	K11a42	K11a45
K11a46	K11a49	K11a50	K11a58	K11a59	K11a61	K11a62	K11a63	K11a64	K11a65
K11a74	K11a75	K11a77	K11a80	K11a81	K11a82	K11a84	K11a85	K11a86	K11a89
K11a90	K11a91	K11a93	K11a95	K11a96	K11a97	K11a98	K11a103	K11a104	K11a108
K11a110	K11a111	K11a117	K11a119	K11a120	K11a121	K11a123	K11a133	K11a134	K11a135
K11a140	K11a141	K11a142	K11a144	K11a145	K11a148	K11a154	K11a159	K11a161	K11a166
K11a167	K11a174	K11a175	K11a176	K11a177	K11a178	K11a179	K11a180	K11a181	K11a182
K11a183	K11a184	K11a185	K11a186	K11a188	K11a190	K11a191	K11a192	K11a193	K11a195
K11a198	K11a199	K11a200	K11a202	K11a203	K11a204	K11a205	K11a206	K11a207	K11a208
K11a210	K11a211	K11a214	K11a218	K11a220	K11a223	K11a224	K11a225	K11a226	K11a228
K11a229	K11a230	K11a234	K11a235	K11a236	K11a238	K11a242	K11a243	K11a245	K11a246
K11a247	K11a249	K11a250	K11a256	K11a258	K11a259	K11a260	K11a263	K11a268	K11a278
K11a279	K11a280	K11a282	K11a286	K11a293	K11a296	K11a299	K11a303	K11a305	K11a306
K11a307	K11a308	K11a309	K11a310	K11a311	K11a313	K11a320	K11a321	K11a323	K11a324
K11a325	K11a330	K11a333	K11a334	K11a335	K11a336	K11a337	K11a339	K11a341	K11a342
K11a343	K11a345	K11a346	K11a355	K11a356	K11a357	K11a358	K11a359	K11a360	K11a361
K11a362	K11a363	K11a364	K11a365	K11a366	K11a367	K12a1	K12a2	K12a9	K12a18
K12a20	K12a22	K12a24	K12a25	K12a28	K12a31	K12a32	K12a37	K12a38	K12a39
K12a54	K12a55	K12a56	K12a78	K12a87	K12a95	K12a96	K12a97	K12a103	K12a104
K12a105	K12a106	K12a110	K12a118	K12a123	K12a125	K12a128	K12a146	K12a147	K12a148
K12a152	K12a153	K12a156	K12a158	K12a159	K12a160	K12a161	K12a165	K12a168	K12a169
K12a172	K12a174	K12a176	K12a178	K12a181	K12a183	K12a195	K12a196	K12a197	K12a204
K12a206	K12a216	K12a217	K12a221	K12a226	K12a229	K12a238	K12a239	K12a241	K12a243
K12a246	K12a247	K12a248	K12a250	K12a251	K12a254	K12a255	K12a257	K12a259	K12a260
K12a270	K12a272	K12a291	K12a297	K12a300	K12a302	K12a303	K12a304	K12a306	K12a307
K12a327	K12a330	K12a331	K12a344	K12a345	K12a353	K12a356	K12a357	K12a358	K12a360
K12a365	K12a369	K12a370	K12a373	K12a375	K12a376	K12a378	K12a379	K12a380	K12a382
K12a384	K12a385	K12a397	K12a398	K12a399	K12a401	K12a404	K12a406	K12a414	K12a421
K12a422	K12a423	K12a424	K12a425	K12a433	K12a436	K12a437	K12a443	K12a444	K12a447
K12a448	K12a454	K12a471	K12a476	K12a477	K12a482	K12a497	K12a498	K12a499	K12a500
K12a501	K12a502	K12a503	K12a504	K12a506	K12a507	K12a508	K12a510	K12a511	K12a512
K12a514	K12a515	K12a517	K12a518	K12a519	K12a520	K12a521	K12a522	K12a526	K12a527
K12a528	K12a532	K12a533	K12a534	K12a535	K12a536	K12a537	K12a538	K12a539	K12a540
K12a541	K12a542	K12a545	K12a549	K12a550	K12a551	K12a552	K12a556	K12a557	K12a561
K12a563	K12a564	K12a565	K12a568	K12a569	K12a573	K12a576	K12a577	K12a579	K12a580
K12a581	K12a582	K12a583	K12a584	K12a585	K12a595	K12a596	K12a597	K12a600	K12a601
K12a605	K12a617	K12a619	K12a628	K12a632	K12a635	K12a636	K12a640	K12a641	K12a643
K12a644	K12a646	K12a648	K12a649	K12a650	K12a651	K12a652	K12a653	K12a657	K12a663
K12a667	K12a669	K12a670	K12a676	K12a677	K12a679	K12a682	K12a683	K12a684	K12a685
K12a686	K12a690	K12a691	K12a693	K12a702	K12a711	K12a713	K12a714	K12a715	K12a716
K12a717	K12a718	K12a719	K12a720	K12a721	K12a722	K12a723	K12a724	K12a725	K12a726
K12a727	K12a728	K12a729	K12a731	K12a732	K12a733	K12a736	K12a738	K12a740	K12a743
K12a744	K12a745	K12a748	K12a749	K12a753	K12a758	K12a759	K12a760	K12a761	K12a762
K12a763	K12a764	K12a770	K12a773	K12a774	K12a775	K12a784	K12a786	K12a789	K12a790
K12a791	K12a792	K12a794	K12a795	K12a796	K12a797	K12a799	K12a800	K12a802	K12a803
K12a805	K12a806	K12a807	K12a808	K12a811	K12a815	K12a816	K12a817	K12a818	K12a820
K12a824	K12a826	K12a827	K12a833	K12a834	K12a838	K12a839	K12a840	K12a842	K12a843
K12a845	K12a847	K12a848	K12a849	K12a850	K12a851	K12a855	K12a858	K12a859	K12a860
K12a880	K12a881	K12a882	K12a883	K12a889	K12a905	K12a911	K12a912	K12a913	K12a916
K12a920	K12a929	K12a937	K12a938	K12a950	K12a952	K12a955	K12a969	K12a970	K12a972
K12a974	K12a975	K12a978	K12a984	K12a991	K12a996	K12a1015	K12a1017	K12a1023	K12a1024
K12a1029	K12a1030	K12a1031	K12a1033	K12a1034	K12a1039	K12a1040	K12a1047	K12a1051	K12a1052
K12a1060	K12a1063	K12a1068	K12a1083	K12a1089	K12a1093	K12a1095	K12a1106	K12a1107	K12a1114
K12a1115	K12a1116	K12a1118	K12a1124	K12a1125	K12a1126	K12a1127	K12a1128	K12a1129	K12a1130
K12a1131	K12a1132	K12a1133	K12a1134	K12a1135	K12a1136	K12a1138	K12a1139	K12a1140	K12a1142
K12a1145	K12a1146	K12a1147	K12a1148	K12a1149	K12a1151	K12a1153	K12a1156	K12a1157	K12a1158
K12a1159	K12a1161	K12a1162	K12a1163	K12a1164	K12a1165	K12a1166	K12a1168	K12a1169	K12a1171
K12a1176	K12a1177	K12a1178	K12a1179	K12a1180	K12a1181	K12a1183	K12a1194	K12a1202	K12a1205
K12a1210	K12a1211	K12a1214	K12a1220	K12a1222	K12a1225	K12a1229	K12a1240	K12a1242	K12a1243
K12a1244	K12a1247	K12a1248	K12a1249	K12a1258	K12a1259	K12a1260	K12a1262	K12a1264	K12a1273
K12a1274	K12a1275	K12a1276	K12a1277	K12a1278	K12a1279	K12a1281	K12a1282	K12a1285	K12a1286
K12a1287	K12a1288								

The second table contains non-alternating CCA knots with the minimal CCA diagram.

8 ₁₉	3, 3, -2	$\{\{8\}, \{6, 8, -12, 2, 14, 16, -4, 10\}\}$
8 ₂₀	3, 2 1, -2	$\{\{8\}, \{4, 8, -12, 2, 14, 16, -6, 10\}\}$
8 ₂₁	2 1, 2 1, -2	$\{\{8\}, \{4, 8, -12, 2, 14, -6, 16, 10\}\}$
9 ₄₂	2 2, 3, -2	$\{\{9\}, \{4, 8, 18, -14, 2, 16, -6, 10, 12\}\}$
9 ₄₃	2 1 1, 3, -2	$\{\{9\}, \{4, 8, 10, -14, 2, 16, 18, -6, 12\}\}$
9 ₄₄	2 2, 2 1, -2	$\{\{9\}, \{4, 8, -12, 2, 16, -6, 18, 10, 14\}\}$
9 ₄₅	2 1 1, 2 1, -2	$\{\{9\}, \{4, 8, 10, -16, 2, 14, 18, -6, 12\}\}$
9 ₄₆	3, 3, -3	$\{\{9\}, \{8, -12, 16, 14, 18, -4, -2, 6, 10\}\}$
9 ₄₇	8* - 2 0	$\{\{9\}, \{6, 8, 10, 16, 14, -18, 4, 2, -12\}\}$
9 ₄₈	2 1, 2 1, -3	$\{\{9\}, \{4, 10, -14, -12, 16, 2, -6, 18, 8\}\}$
9 ₄₉	-2 0 : -2 0 : -2 0	$\{\{9\}, \{6, -10, -14, 12, -16, -2, 18, -4, -8\}\}$
10 ₁₅₀	6* - 2.2.2.2 0	$\{\{10\}, \{6, 10, 16, 20, 14, 2, -18, 4, 8, -12\}\}$
K11n8	6*2.2 1 0 : -3 0	$\{\{11\}, \{4, 8, 16, 20, 2, -18, 6, 22, -12, -10, 14\}\}$
K11n115	6*2 - 3.2 : 2 0	$\{\{11\}, \{6, 12, 16, 22, -18, -20, 2, 8, 4, -10, 14\}\}$
K11n123	6* - 3.2.2.2 0	$\{\{11\}, \{6, 10, 16, 22, 18, 2, -20, 8, 4, -14, -12\}\}$
K11n124	6*2 - 2.2.2.2 0	$\{\{11\}, \{6, -10, 14, 20, -2, 18, 4, 22, 12, 8, 16\}\}$
K11n143	6* - 2.2 - 2.2 0.2 0	$\{\{11\}, \{6, 12, -16, 22, -18, 2, 20, -4, -8, 14, 10\}\}$
K11n157	9* - 3	$\{\{11\}, \{6, 18, 16, 12, 4, 2, -20, -22, 10, 8, -14\}\}$
K12n147	-2 - 1 - 1, 2 1 1, 2 1 1	$\{\{12\}, \{4, 14, 18, 16, -12, -22, 2, 24, 20, 6, -10, -8\}\}$

The knot 10₁₅₁ = (2 1, 2) (2 1, -2) (Fig. 2) is the example of a CCA knot without minimal CCA diagram. Its non-minimal diagram 6*2 - 1 - 1.2 : 2 0 with the DT code $\{\{11\}, \{-4, 10, 16, 20, 2, -22, 18, 8, 12, 6, 14\}\}$ is CCA.

For all the other knots with $n \leq 12$ crossings we don't know are they CCA or not. If they are, they can have only non-minimal CCA diagrams.

For all knots K up to $n = 11$ crossings T. Abe [2] proved that $alt(K) \leq 1$. Hence, we need to consider only knots with $n = 12$ crossings.

We computed alternation numbers from all minimal diagrams, and concluded that all knots K with $n = 12$ crossings have the alternation number $alt(K) \leq 1$, except the knots K12n426, K12n706, K12n801, K12n835, K12n838, K12n888 with the alternation number 2.

REFERENCES

- [1] A. Kawauchi, On alternation numbers of links, <http://www.sci.osaka-cu.ac.jp/~kawauchi/altnumber.pdf>
- [2] T. Abe, An estimation of the alternation number of a torus knot, J. Knot Theory Ramifications **18**, 3 (2009) 363-379.
- [3] S. V. Jablan, R. Sazdanović, *LinKnot- Knot Theory by Computer*. World Scientific, New Jersey, London, Singapore, 2007, <http://math.ict.edu.rs/>.

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